Elastic Analogies

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In a variety of situations, including the numerical solution of partial differential equations, it is useful to be able to exploit analogies between the equations of one discipline and those of another. Here we summarize an analogy between the equations of linear elasticity and the common equations of mathematical physics. One consequence of this analogy is that standard structural analysis computer programs can be used, without modification, to solve a variety of scalar field problems. The analogy is particularly useful in fluid-structure interaction problems, where the coupling of an elastic structure with an acoustic field is the motivation for solving wave, Helmholtz, and Laplace equations with a structural code.

Classical Equations of Mathematical Physics

Many of the classical equations of mathematical physics can be viewed as special cases of the general equation

$$\nabla^2\phi + g = a\ddot{\phi} + b\dot{\phi}, \tag{1}$$

where ∇^2 is the Laplacian operator, dots denote partial time differentiation, the functions g, a, and b are, in general, position-dependent, and the unknown scalar function ϕ depends on both position and time.

Special cases of Eq. 1 arise in such diverse applications as heat conduction, acoustics, electrical and magnetic potential problems, torsion of prismatic bars, potential fluid flow, and seepage through porous media. Several common special cases are listed below:

$$\begin{aligned}
\text{Laplace's equation:}\quad & \nabla^2\phi = 0 && (2)\\
\text{Poisson's equation:}\quad & \nabla^2\phi + g = 0 && (3)\\
\text{wave equation:}\quad & \nabla^2\phi = \ddot{\phi}/c^2 && (4)\\
\text{heat equation:}\quad & k\nabla^2\phi + q = \rho c\dot{\phi} && (5)\\
\text{telegraph equation:}\quad & \partial^2\phi/\partial x^2 = LC\ddot{\phi} + RC\dot{\phi} && (6)\\
\text{Helmholtz equation:}\quad & \nabla^2\phi + k^2\phi = 0. && (7)
\end{aligned}$$

The Helmholtz equation is the time-harmonic form of the wave equation. That is, for time-harmonic motion, $\phi = \phi_0\cos(\omega t)$, and the wave equation simplifies to the Helmholtz equation, where ϕ_0 is the amplitude of the sine wave, and $k = \omega/c$ is called the wave number. The Helmholtz equation is sometimes referred to as the reduced wave equation.

These classical equations of mathematical physics arise in many physical situations. For example, Laplace's equation (the potential equation) arises in incompressible fluid flow (where the fundamental unknown is the velocity potential), gravitational potential problems, electrostatics, magnetostatics, steady-state heat conduction, and the torsion problem of elasticity. The Poisson equation arises in steady-state heat conduction (with distributed sources) and torsion of prismatic bars in elasticity. The wave and Helmholtz equations arise, respectively, in transient and time-harmonic elastic vibrations (strings, bars, and membranes), acoustics, and electromagnetics. The heat equation arises in heat conduction and other diffusion processes.

Summary of Equations of Elastodynamics

Consider a general three-dimensional elastic body acted upon by forces. The dynamic equilibrium (momentum) equations for this body are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x = \rho \ddot{u} \tag{8}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y = \rho \ddot{v} \tag{9}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho \ddot{w}, \tag{10}$$

where σ_{ij} is the jth component of stress acting on a surface with normal in the ith coordinate direction, ρ is the material density, f_i is the ith component of the body force per unit mass (e.g., gravity), and u, v, w are the three Cartesian components of displacement. From the balance of moments, the stress components are symmetric: $\sigma_{ij} = \sigma_{ji}$.

The strain-displacement equations are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{11}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{12}$$

$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{13}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \tag{14}$$

The components of strain in Eq. 11 are referred to as the normal (or direct) strains. The components of strain in Eqs. 12–14 are the shear strains. The strains γ_{ij} are the engineering shear strains. Like the stresses, the strain components are also symmetric: $\varepsilon_{ij} = \varepsilon_{ji}$.

Stresses are related to strains by generalized Hooke's law, which states that each stress component is a linear combination of all the strain components. For general anisotropy, the six (unique) components of stress are thus related to the six components of strain by at most 21 experimentally-measured elastic constants. For an isotropic material (for which all directions are equivalent), the number of independent elastic constants reduces to two, and Hooke's law can be written in the form

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{array}\right\} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 2\varepsilon_{xy}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{xz}\end{array}\right\}. \tag{15}$$

where λ and μ are the Lamé elastic constants (where $\mu = G$ is the shear modulus).

The Lamé constants λ and μ can be expressed in terms of the more familiar engineering constants (Young's modulus E, Poisson's ratio ν, and shear modulus G) as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{16}$$

or, inversely,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{17}$$

In terms of the engineering constants, Hooke's law can also be written in inverse form as

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} \tag{18}$$

$$\varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} \tag{19}$$

$$\varepsilon_{zz} = -\nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E} \tag{20}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\sigma_{xy}}{G} \tag{21}$$

$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\sigma_{yz}}{G} \tag{22}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\sigma_{xz}}{G}. \tag{23}$$

The substitution of Hooke's law and the strain-displacement equations into the momentum equations yields the Navier equations of elasticity, which are partial differential equations of motion for the displacements. For example, the x-component of the Navier equations is

$$\left(\frac{\lambda+2\mu}{\mu}\right)u_{,xx} + u_{,yy} + u_{,zz} + \left(\frac{\lambda+\mu}{\mu}\right)(v_{,xy} + w_{,xz}) + \frac{\rho}{\mu}f_x = \frac{\rho}{\mu}\ddot{u}, \tag{24}$$

where commas denote partial differentiation.

The Analogy

Eq. 24 reduces to Eq. 1 if ϕ is represented by u (the x-component of displacement) and

$$\frac{\lambda + 2\mu}{\mu} = 1 \tag{25}$$

$$v \equiv w \equiv 0 \tag{26}$$

$$\frac{\rho}{\mu} = a \tag{27}$$

$$\frac{\rho}{\mu} f_x = g - b\dot{\phi}. \tag{28}$$

Thus, if the shear modulus μ for the "elastic" model is μ_e (which can, as will be seen, be selected arbitrarily), the other material constants are

$$\lambda_e = -\mu_e \tag{29}$$

$$\rho_e = \mu_e a, \tag{30}$$

and the body force per unit volume is

$$\rho_e f_x = \mu_e (g - b\dot{\phi}). \tag{31}$$

The two terms in this equation represent, respectively, a gravitational type of force and a distributed dashpot. Thus, the total force F_x acting at a point of the elastic model to which the volume V has been assigned is

$$F_x = \mu_e g V - (\mu_e b V)\dot{\phi}, \tag{32}$$

where the dashpot constant is $\mu_e b V$. If the forcing function g in Eqs. 1 and 32 is a constant (independent of position), it may be specified conveniently by applying to the elastic model a gravitational field whose gravitational constant g_0 satisfies

$$\rho_e g_0 = \mu_e g. \tag{33}$$

Eq. 29 implies that the Young's modulus E and Poisson's ratio ν must both be infinite. Instead, we choose

$$E_e = \alpha \mu_e \quad (\alpha \gg 1) \tag{34}$$

$$\nu_e = \frac{E_e}{2\mu_e} - 1 \approx \frac{\alpha}{2}. \tag{35}$$

Eqs. 16, 17, 34, and 35 indicate that α should be chosen large enough so that $\alpha + 1$ is indistinguishable numerically from α (e.g., $\alpha = 10^{20}$).

To summarize, the scalar equation, Eq. 1, can be solved with an elastic model having the material properties

$$E_e = \alpha \mu_e, \quad \nu_e = \frac{\alpha}{2}, \quad \rho_e = \mu_e a \quad (\alpha \gg 1). \tag{36}$$

Although the preceding derivation chose u, the x-component of displacement, to represent the scalar field variable ϕ (with the other two components v and w constrained to zero), any of the three Cartesian displacement components could be used. The shear modulus μ_e in Eq. 36 is arbitrary, since all dimensional material constants, as well as the load given by Eq. 32, are proportional to it. Consequently, it is convenient in some cases to choose $\mu_e = 1$.

Axisymmetric Problems

The solution of three-dimensional problems in cylindrical coordinates follows the same approach as in Cartesian coordinates, except that the z-component of displacement (which is the same in both Cartesian and cylindrical coordinates) is the only component which may be used to represent the scalar field variable ϕ. Neither the r-component nor the θ-component of the Navier equations can be specialized to yield Eq. 1.

Two-Dimensional Problems

The development of the analogy in two-dimensional (plane stress) elasticity is similar to that in three dimensions but results in different values for the material constants E and ν. It is convenient to start from Hooke's law (Eq. 15) with $\lambda_e = -\mu_e$ (Eq. 29), from which it follows that the 6×6 material matrix $\mathbf{G}$ relating the six stress components to the corresponding strain components is the coefficient matrix in

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{array} \right\} = \mu_e \left[\begin{array}{rrrrrr} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ v_{,z} + w_{,y} \\ w_{,x} + u_{,z} \end{array} \right\}. \tag{37}$$

The two-dimensional counterpart of this constitutive equation is

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \mu_e \left[\begin{array}{rrr} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{array} \right\}. \tag{38}$$

Thus, the two-dimensional material matrix is

$$\mathbf{G} = \mu_e \left[\begin{array}{rrr} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \tag{39}$$

If this matrix is compared to the material matrix for plane stress,

$$\mathbf{G} = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right], \tag{40}$$

the material constants for the elastic model are

$$E_e = \beta \mu_e, \quad \nu_e = \frac{1}{2}\beta - 1, \quad \rho_e = \mu_e a \quad (\beta \ll 1), \tag{41}$$

where β must be small compared to unity but not so small that $1 + \beta$ is indisguishable numerically from unity (e.g., $\beta = 10^{-5}$).

To summarize, the two-dimensional form of Eq. 1 can be solved with a two-dimensional plane stress elastic model whose material constants satisfy Eq. 41. In Cartesian coordinates, either the x- or y-component of displacement may be chosen to represent the scalar field variable ϕ, with the other component constrained to zero everywhere. The shear modulus μ_e in Eq. 41 is arbitrary, as it is in three dimensions, and hence may be conveniently taken as unity.

Boundary Conditions

Most boundary conditions likely to be encountered in connection with Eq. 1 will likely be special cases of the general form

$$a_1 \frac{\partial \phi}{\partial n} + a_2 \phi + a_3 \dot{\phi} + a_4 \ddot{\phi} + a_5 = 0, \tag{42}$$

where n is the outward normal. To relate this boundary condition to the elastic analogy, we first examine the normal derivative.

Let $\mathbf{n}$ denote the unit outward normal from the domain at a surface point. The goal is to evaluate the normal derivative $\partial \phi / \partial n$. When ϕ is replaced with its elastic analog u (the x-component of displacement),

$$\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n} = u_{,x} n_x + u_{,y} n_y + u_{,z} n_z. \tag{43}$$

The constitutive equation, Eq. 37, and the constraint, Eq. 26, yield

$$\frac{\partial u}{\partial n} = (\sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z)/\mu_e, \tag{44}$$

where the parenthetical expression is equal to the x-component of the stress vector $\mathbf{t}^{(n)}$ acting on a surface with outward normal $\mathbf{n}$. Hence,

$$\frac{\partial u}{\partial n} = \frac{t_x^{(n)}}{\mu_e} = \frac{F_x}{\mu_e A}, \tag{45}$$

where F_x is the x-component of the force applied to a particular point (on the surface with outward normal $\mathbf{n}$) to which the area A has been assigned. Thus,

$$F_x = \mu_e A \frac{\partial u}{\partial n}. \tag{46}$$

In other words, to enforce the outward normal derivative $\phi_{,n}$ at a surface point, one applies a "load" at that point in the elastic model equal to $\mu_e A \phi_{,n}$. A positive force F_x corresponds to a positive outward normal derivative.

From Eq. 46, the boundary condition, Eq. 42, is enforced by applying a "load" to each boundary point (to which the area A has been assigned) given by

$$F_x = -\frac{\mu_e A}{a_1} (a_2 \phi + a_3 \dot{\phi} + a_4 \ddot{\phi} + a_5) \quad (a_1 \neq 0). \tag{47}$$

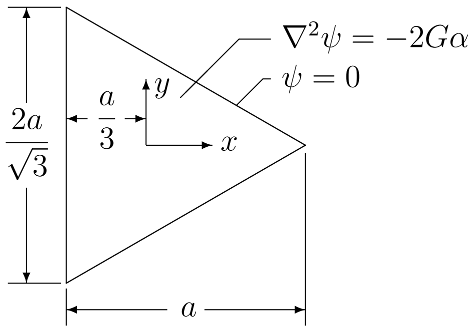

Figure 1: Torsion of Triangular Prism.

The a_2 term is analogous to a scalar spring of constant $\mu_e A a_2/a_1$ connected between the point and ground. The a_3 term is analogous to a scalar dashpot of constant $\mu_e A a_3/a_1$ connected between the point and ground. The a_4 term is analogous to an added mass of value $\mu_e A a_4/a_1$. The a_5 term is a time-independent force given by $-\mu_e A a_5/a_1$. For physically meaningful boundary conditions, the boundary condition, Eq. 42, must be specified so that a_1, a_2, a_3, and a_4 all have the same sign. Otherwise, under the analogy, the result would be a negative spring, dashpot, or mass.

As expected, the special case of the Neumann boundary condition ($\phi_{,n} = 0$) corresponds to the traction-free boundary in elasticity (and hence is a natural boundary condition in displacement-based finite element methods). The Dirichlet condition ($\phi = \phi_0$) is implemented merely by enforcing the desired value as a “displacement” boundary condition.

Example: Torsion of Prismatic Bars

A simple example involving the torsion of prismatic bars can illustrate the use of the structural analogy. The stress distribution over a non-circular cross-section of a twisted bar can be determined by finding the stress function $\psi(x, y)$ which satisfies the two-dimensional Poisson equation

$$\nabla^2\psi = -2G\alpha \tag{48}$$

in the cross-section and is zero on the boundary, where G is the shear modulus, and α is the angle of twist per unit length. The stresses of interest are obtained by differentiation:

$$\sigma_{zx} = \frac{\partial\psi}{\partial y}, \quad \sigma_{zy} = -\frac{\partial\psi}{\partial x}. \tag{49}$$

The torsional constant J for the cross-section is given by

$$J = \frac{2}{G\alpha}\int_A \psi \, dA. \tag{50}$$

The specific cross-section considered here is the equilateral triangle of altitude a shown in Fig. 1. For this domain, Eq. 48 can be solved in closed form to yield

$$\psi = -G\alpha[(x^2 + y^2)/2 - (x^3 - 3xy^2)/2a - 2a^2/27]. \tag{51}$$

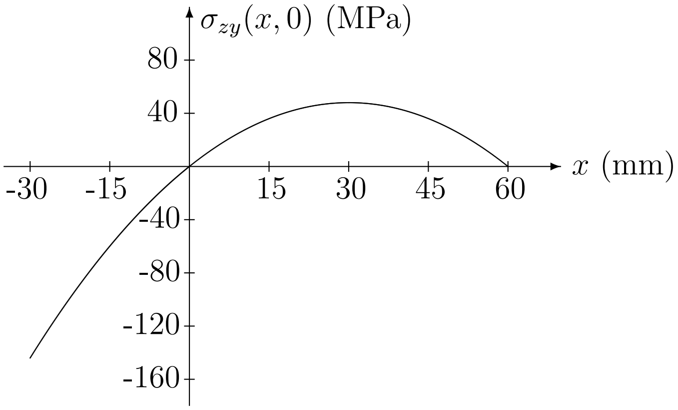

Figure 2: Shear Stress σ_{zy} Along $y = 0$.

Along the x-axis, the stresses are obtained from Eq. 49 as

$$\sigma_{zx}(x,0) = 0, \quad \sigma_{zy}(x,0) = 3G\alpha(2ax/3 - x^2)/(2a). \tag{52}$$

The maximum stress occurs at the middle of the sides of the triangle ($x = -a/3$); hence,

$$(\sigma_{zy})_{\max} = G\alpha a/2. \tag{53}$$

The torsional constant is obtained by substituting Eq. 51 into Eq. 50:

$$J = \frac{a^4}{15\sqrt{3}}. \tag{54}$$

To illustrate the analogy, we perform a numerical experiment using a plane stress finite element computer program. The following parameters were chosen:

$a = 0.09$ m
$G = 80$ GPa
$\alpha = 0.04$ m^{-1}.

For these parameters, the shear stress $\sigma_{zy}(x,0)$, Eq. 52, is plotted in Fig. 2.

In the finite element model, the upper half of the triangular cross-section was modeled, although symmetry would require modeling only one-sixth of the triangle. Nine elements were used, as shown in Fig. 3, including one degenerate quadrilateral in the lower right corner of the mesh. The element used was the standard two-dimensional, eight-node, quadratic, isoparametric, plane stress membrane element available in many finite element structural analysis computer programs. A 3 × 3 array of Gauss numerical integration points was selected for each element.

According to Eq. 41, the elastic material properties of each plane stress finite element were chosen as

$$\mu_e = G_e = 1, \quad E_e = 10^{-5}. \tag{55}$$

The Poisson equation for torsion, Eq. 48, is a special case of the general equation, Eq. 1, with

$$g = 2G\alpha, \quad a = b = 0. \tag{56}$$

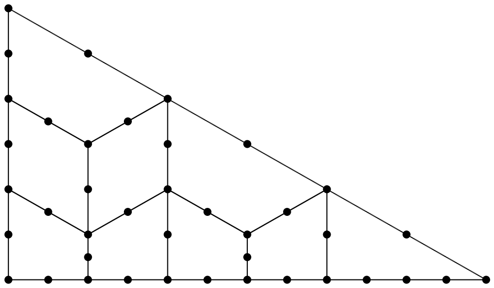

Figure 3: Finite Element Mesh for Triangular Cross-Section.

Hence, from the first term of Eq. 32,

$$F_x = 2G\alpha G_e V. \tag{57}$$

Note the distinction between G_e, the "shear modulus" specified for the elastic model under the analogy, and G, the actual shear modulus for the triangular bar. The body force in Eq. 57, which is proportional to the volume assigned to each point, is most conveniently specified in a finite element model as a gravitational load. Since a gravitational field applies the load $\rho_e g_0 V$ to each point, where ρ_e is the element mass density, and g_0 is the acceleration due to gravity, it follows that ρ_e and g_0 must be specified so that

$$\rho_e g_0 = 2G\alpha G_e = 2(80)(0.04)(1) = 6.4. \tag{58}$$

Thus, since both constants are otherwise arbitrary,

$$\rho_e = 1, \quad g_0 = 6.4. \tag{59}$$

The element thickness is arbitrary since both the plane stress "stiffness" matrix and the "load" are proportional to it.

The actual stresses σ_{zx} and σ_{zy} given by Eq. 49 may be obtained using the two-dimensional stress-strain law for plane stress elasticity, Eq. 38:

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \left[\begin{array}{rrr} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{array} \right\}, \tag{60}$$

where the elastic constants of Eq. 55 have been used, and v (the y-component of displacement) is everywhere constrained to zero. Hence, since ψ is represented by u,

$$\left\{ \begin{array}{l} \sigma_{xx} = u_{,x} = \psi_{,x}, \\ \sigma_{yy} = -u_{,x} = -\psi_{,x}, \\ \sigma_{xy} = u_{,y} = \psi_{,y}. \end{array} \right. \tag{61}$$

That is, the stresses designated by the plane stress finite element program as σ_{xy} and σ_{yy} correspond, respectively, to σ_{zx} and σ_{zy}, the torsional shear stresses of interest.

The finite element calculation of the torsional constant using Eq. 50 requires the numerical integration of the stress function ψ over the cross-section. For a finite element mesh having

N grid points, this integral can be approximated by

$$J = \frac{2}{G\alpha} \sum_{i=1}^{N} \psi_i A_i, \tag{62}$$

where ψ_i is the value of ψ (the solution) at Point i, and A_i is the area assigned to that point. The area A_i can be obtained from the applied "load" vector, since a gravitational field applies a force F_i at Point i given by

$$F_i = \rho_e g_0 t A_i, \tag{63}$$

where t is the element thickness. Thus,

$$J = \frac{2}{G\alpha\rho_e g_0 t} \sum_{i=1}^{N} \psi_i F_i, \tag{64}$$

where the summation is equal to the dot product of the finite element solution vector and the applied force vector (twice the "strain energy"). Since, by symmetry, only half the cross-section was modeled, the result obtained with a half-model would have to be doubled to account for the unmodeled half.

For this numerical example, the torsional constant obtained by the finite element program was 252.2 cm^4, which differs by about 0.12% from the exact value of 252.5 cm^4 calculated using Eq. 54.