

The George Washington University  
Washington, D.C.

**ApSc 213 – Analytical Methods in Engineering III**  
**Partial Differential Equations**

Fall 2009 – Main Campus

- References:** *Partial Differential Equations with Fourier Series and Boundary Value Problems*, Second Edition, by N.H. Asmar (Pearson Prentice Hall, 2004), ISBN 0131480960; *Fourier Series and Boundary Value Problems*, Seventh Edition, by J.W. Brown and R.V. Churchill (McGraw-Hill, 2006), ISBN 0073051934.
- Notes:** *Analytical Solution of Partial Differential Equations* by G.C. Everstine
- Instructor:** Gordon C. Everstine, <http://gwu.geverstine.com>  
[geversti@gwu.edu](mailto:geversti@gwu.edu), 301-977-0936
- Schedule:** Wednesdays, Sep. 2 – Dec. 16, 3:30 p.m. – 6:00 p.m.  
No class: Dec. 9 (reading day)  
Mid-Term Exam: Oct. 21  
Final Exam: Dec. 16
- Description:** Analytical techniques for solution of boundary-initial-value problems in engineering; wave propagation, diffusion processes, and potential distributions.
- Objectives:** To understand the derivation and applicability of the classical partial differential equations of engineering; to increase knowledge of the nature of solutions of equations of different types; to learn how to solve various equations analytically.
- Grading:** Assignments 1/3, mid-term exam 1/3, final exam 1/3. All graded work must be completed in accordance with the GW Code of Academic Integrity (<http://www.gwu.edu/~ntegrity/code.html>). Students are encouraged to discuss the meaning of assignments and general approaches and strategies for handling those assignments, but it is not acceptable to share solutions.

**Course Outline**

1. Review of notation and integral theorems; the divergence theorem; Green's theorems
2. Derivation of wave and heat equations; elastodynamics; initial conditions and boundary conditions; uniqueness; classification of partial differential equations
3. Fourier series; expansions in orthogonal functions; generalized Fourier series; completeness
4. Problems in Cartesian coordinates; transient and steady-state problems; nonhomogeneous equations
5. Sturm-Liouville systems; orthogonality of eigenfunctions
6. Orthogonal curvilinear coordinates
7. Problems in cylindrical coordinates; Bessel's equation
8. Problems in spherical coordinates; Legendre's equation

ApSc 213 – Assignment 1

1. For the scalar function  $f = x^3yz + x^2z + z^2y^2$ , find (a) the rate of change of  $f$  in the direction given by the vector  $\mathbf{A} = x^2y\mathbf{e}_x + xz\mathbf{e}_y + xyz\mathbf{e}_z$  at the point  $(1, 1, 1)$ , and (b)  $\nabla^2 f$ .
2. The vector  $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$  is referred to as the *position vector* in Cartesian coordinates. Find  $\nabla \cdot \mathbf{r}$  and  $\nabla \times \mathbf{r}$ .
3. Evaluate the surface integral of the normal component of the vector  $\mathbf{F} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$  over the closed surface of the cube bounded by the planes  $x = \pm 1$ ,  $y = \pm 1$ ,  $z = \pm 1$ .
4. Evaluate the surface integral of the normal component of the vector field

$$\mathbf{F} = yz\mathbf{e}_x + xz\mathbf{e}_y + xy\mathbf{e}_z$$

over the closed surface of the 3-D region bounded above by  $z = 1$  and below by  $z = x^2 + y^2$ .

5. A uniform (constant) pressure is applied to the surface of a body of arbitrary shape. Prove that the resultant moment  $\mathbf{M}$  of this distribution of surface loading is zero.
6. An arbitrary body of volume  $V$  is fully submerged in a fluid of uniform weight density  $\gamma$ . Given that the hydrostatic pressure  $p$  is  $\gamma z$  at a fluid depth  $z$  (positive down), compute the resultant force  $\mathbf{F}$  acting on the body.
7. Given a general closed surface  $S$  for which the position vector  $\mathbf{r}$  and normal  $\mathbf{n}$  are known at every point, derive a formula for the volume enclosed by  $S$ . Verify your relation for the special case of a sphere.

ApSc 213 – Assignment 2

1. A function  $f(x)$  is defined as *even* if  $f(-x) = f(x)$  for all  $x$ . A function  $f(x)$  is defined as *odd* if  $f(-x) = -f(x)$  for all  $x$ . Prove, using other than geometrical arguments, the following properties of even and odd functions:

- (a) The product of two even functions is even.
- (b) The product of an even and an odd function is odd.
- (c) The product of two odd functions is even.
- (d)  $f(x)$  even implies

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

- (e)  $f(x)$  odd implies

$$\int_{-a}^a f(x) dx = 0.$$

2. Consider the general 3-D heat conduction problem in which no heat generation is present in the body, and only the heat flux  $k\partial T/\partial n$  is specified on the boundaries of the body, where  $k$  is a constant. Determine a condition on the distribution of  $\partial T/\partial n$  over the surface of the body so that a steady-state condition ( $\partial T/\partial t=0$ ) can be reached. Interpret this condition physically. Hint: Apply the divergence theorem to the applicable PDE.
3. A large inhomogeneous slab consisting of two homogeneous layers with different thermal properties is heated to a certain temperature  $T_0$ , and then cooled by having its faces held at zero temperature starting from the time  $t = 0$ . Assuming that the faces of the slab are at  $x = 0$  and  $x = a_1 + a_2$  (where  $a_1$  and  $a_2$  are the thicknesses of the two layers), formulate the problem. That is, specify the equations and conditions which must be satisfied by the temperature distribution in the slab. Hint: The key issue here is what happens at the interface.

ApSc 213 – Assignment 3

1. The transverse displacement of a string stretched between two fixed end points obeys the one-dimensional wave equation,  $u_{xx} = u_{tt}/c^2$ , where  $c$  is a constant. Let the length of the string at rest be  $L$ , and assume the string is released from rest at  $t = 0$  from the initial position  $f(x)$ . That is, the boundary and initial conditions are  $u(0, t) = u(L, t) = 0$ ,  $u(x, 0) = f(x)$ , and  $u_t(x, 0) = 0$ . Use separation of variables to solve for  $u(x, t)$ .
2. Consider the one-dimensional, steady-state heat conduction in a nonhomogeneous rod of length  $L$ . No internal heat sources are present. Assume that the rod lies on the  $x$ -axis between  $x = 0$  and  $x = L$ . The conductivity  $k$  in the rod is given by  $k = k_0/(1 + x/L)$ , where  $k_0$  is a positive constant. What differential equation does the temperature  $T$  satisfy in the rod? Given that  $T = T_1$  at  $x = 0$  and that  $T = T_2$  at  $x = L$ , find the temperature distribution along the rod.
3. If  $S$  is the surface with outward unit normal  $\mathbf{n}$  bounding the volume  $V$ , and

$$-\oint_S \rho \mathbf{n} \cdot \mathbf{v} dS = \int_V \frac{\partial \rho}{\partial t} dV$$

for all  $V$ , where  $\rho$  is a scalar function of both space and time, find the differential equation resulting from the above relation.

ApSc 213 – Assignment 4

1. Prove that, over  $(0, 2\pi)$ , the following is an orthonormal set:

$$1/\sqrt{2\pi}, (\cos x)/\sqrt{\pi}, (\sin x)/\sqrt{\pi}, (\cos 2x)/\sqrt{\pi}, (\sin 2x)/\sqrt{\pi}, \dots$$

2. Find (a) the Fourier sine series and (b) the Fourier cosine series representing the function  $f(x) = e^x$  in the interval  $(0, \pi)$ . Since there are lots of approximations to  $e^x$  in  $(0, \pi)$ , use the series approximations with fundamental period  $2\pi$  (i.e.,  $L = \pi$ ). (c) Find the complete Fourier series for  $f$  in  $(-\pi, \pi)$ . (d) Make plots of the sine and cosine series obtained in (a) and (b) for different numbers of terms (e.g., 2, 5, 10, 20, 40), and indicate which of the two series you prefer to represent  $e^x$  in  $(0, \pi)$ .

ApSc 213 – Assignment 5

1. Find the solution  $T(x, t)$  to the following one-dimensional heat conduction problem:

$$KT_{xx} = T_t, \quad 0 < x < L$$

$$T(0, t) = T_0, \quad T(L, t) = 2T_0, \quad T(x, 0) = T_0(1 + x/L),$$

where  $K$  and  $T_0$  are constants.

2. Given the 3-D rectangular solid with sides of length  $a$ ,  $b$ , and  $c$  in the  $x$ ,  $y$ , and  $z$  directions, respectively, find the function  $T(x, y, z)$  in the interior of the solid when  $\nabla^2 T = 0$ , and  $T = 0$  for  $x = 0$ ,  $\partial T/\partial x = 0$  for  $x = a$ ,  $\partial T/\partial y = 0$  for  $y = 0$  and  $y = b$ ,  $T = 0$  for  $z = 0$ , and  $T = f(x, y)$  for  $z = c$ .
3. Given the 3-D rectangular solid with sides of length  $a$ ,  $b$ , and  $c$  in the  $x$ ,  $y$ , and  $z$  directions, respectively, find the function  $T(x, y, z, t)$  when  $K\nabla^2 T = \partial T/\partial t$ ,  $T(x, y, z, 0) = 0$ ,  $\partial T/\partial x + h_1 T = 0$  for  $x = 0$ ,  $T = 0$  for  $x = a$ ,  $\partial T/\partial y = 0$  for  $y = 0$  and  $y = b$ ,  $\partial T/\partial z = 0$  for  $z = 0$ , and  $\partial T/\partial z + h_2 T = 0$  for  $z = c$ , where  $h_1$  and  $h_2$  are constants.

ApSc 213 – Assignment 6

1. Solve the following boundary-initial value problem for  $T(x, y, t)$ :

$$K\nabla^2 T = \frac{\partial T}{\partial t} \quad (0 < x < a, \quad 0 < y < b)$$

$$T + \frac{\partial T}{\partial x} = 0 \text{ for } x = 0, \quad \frac{\partial T}{\partial x} = 0 \text{ for } x = a, \quad T = 0 \text{ for } y = 0 \text{ and } y = b, \quad T(x, y, 0) = 1$$

2. Show that the general equation

$$a(x)y''(x) + b(x)y'(x) + [c(x) + \lambda d(x)]y(x) = 0$$

can be put in the form of a Sturm-Liouville equation, where  $a(x)$ ,  $b(x)$ ,  $c(x)$ , and  $d(x)$  are arbitrary functions. Hint: Put both the given equation and the Sturm-Liouville equation in the form  $y'' + \dots$ .

ApSc 213 – Assignment 7

1. Consider spherical coordinates  $(r, \theta, \phi)$  defined by

$$\begin{cases} x = r \sin \theta \cos \phi, \\ y = r \sin \theta \sin \phi, \\ z = r \cos \theta. \end{cases}$$

- (a) Show that this system of coordinates is orthogonal (by verifying that the vectors  $\mathbf{U}_i$  tangent to the coordinate curves are mutually orthogonal).
- (b) Obtain for this system the quantities  $ds^2$ ,  $dV$ ,  $\nabla f$ ,  $\nabla \cdot \mathbf{A}$ ,  $\nabla \times \mathbf{A}$ , and  $\nabla^2 f$ .
- (c) Give a geometrical interpretation to the three coordinate curves.
2. Find the volume  $V$  cut from the spherical solid  $r < a$  by the cone  $\theta < \alpha$ . Evaluate  $V$  for the special cases  $\alpha = \pi/3$ ,  $\alpha = \pi/2$ , and  $\alpha = \pi$ .
3. Consider the homogeneous infinite region  $r > a$ , bounded on the inside by the spherical cavity  $r = a$ .
- (a) Find the steady-state temperature  $T(r, \theta, \phi)$  when  $\nabla^2 T = 0$  for  $r > a$ ,  $T = T_0$  at  $r = a$ , and  $T \rightarrow 0$  for  $r \rightarrow \infty$ , where  $T_0$  is a constant. Hint: Consider first the symmetry of the problem.
- (b) What is the total rate of heat conduction in the radial direction?

ApSc 213 – Assignment 8

1. Along the circumference of the unit circle  $r = 1$ , a solution  $T$  of Laplace's equation is required to take on the value unity when  $0 < \theta < \pi$  and the value zero when  $\pi < \theta < 2\pi$ . Determine an expression for  $T$  valid when  $r < 1$ .
2. What is the temperature at the center of a homogeneous circular disc whose average boundary temperature is  $72^\circ\text{F}$ ? The disc contains no internal heat sources.
3. Derive the solution  $T(r, \theta)$  to Laplace's equation in the 2-D domain exterior to the circle  $r = a$ , where  $T(a, \theta) = f(\theta)$ , and  $T$  remains finite for large  $r$ .
  - (a) What is the average temperature at fixed  $r$ ?
  - (b) What is the temperature  $T_\infty$  at infinity? Interpret this value physically.
  - (c) What is  $T(r, \theta)$  for the special case  $f(\theta) = T_0$ ?
4. Find the function  $T(r, \theta)$  which is harmonic in the 2-D annulus  $a < r < 2a$  and for which  $T(a, \theta) = 1$  ( $0 < \theta < \pi$ ),  $T(a, \theta) = 0$  ( $\pi < \theta < 2\pi$ ),  $T(2a, \theta) = 0$  ( $0 < \theta < \pi$ ),  $T(2a, \theta) = 1$  ( $\pi < \theta < 2\pi$ ).
5. Determine the solution  $T(r, \theta)$  for the 2-D problem  $\nabla^2 T = 0$  ( $a < r < 2a$ ,  $0 < \theta < \pi$ ),  $T(a, \theta) = 1$ ,  $T(2a, \theta) = 0$ ,  $T(r, 0) = T(r, \pi) = 0$ .