The George Washington University
Washington, D.C.

ApSc 212 – Analytical Methods in Engineering II (Linear Algebra)

Spring 2010 – Main Campus

Reference: *Linear Algebra and Its Applications*, fourth edition, by Gilbert Strang (Thomson Brooks/Cole, 2006), ISBN 9780030105678.

Notes: *Applications of Linear Algebra* by G.C. Everstine

Instructor: Gordon C. Everstine, `http://gwu.geverstine.com`
geversti@gwu.edu, 301-977-0936

Schedule: Mondays, Jan. 11 – May 3, and Wednesday, April 28 (designated Monday), 6:10 p.m. – 8:40 p.m.
No class: Jan. 18, Feb. 15, Mar. 15
Mid-Term Exam: March 29
Final Exam: May 3

Description: Algebraic methods appropriate to the solution of engineering computational problems; linear vector spaces, matrices, systems of linear equations, eigenvalues and eigenvectors, quadratic forms.

Objectives: To understand algebraic methods used in the solution of engineering computational problems; to apply some of the algorithms discussed by writing simple computer programs; to appreciate some of the issues involved in commercial engineering software.

Grading: Assignments 1/3, mid-term exam 1/3, final exam 1/3. All graded work must be completed in accordance with the GW Code of Academic Integrity (`http://www.gwu.edu/~ntegrity/code.html`). Students are encouraged to discuss the meaning of assignments and general approaches and strategies for handling those assignments, but it is not acceptable to share solutions and computer codes.

Course Outline

1. Systems of linear equations; Gaussian elimination; operation counts; partial pivoting; LU factorization; determinants; iterative methods
2. Vector spaces; rectangular systems; linear independence; pseudoinverses; linear transformations; orthogonality; projections
3. Change of basis; index notation and tensors; examples
4. Least squares problems; fitting; Gram-Schmidt orthogonalization; QR factorization
5. Fourier series; generalized Fourier series; expansions using polynomial basis
6. Eigenvalue problems; applications to dynamical systems; properties; orthogonality; power iteration; similarity transformations; positive definite matrices; applications to structural dynamics and differential equations

ApSc 212 – Assignment 1

1. When is an upper triangular matrix nonsingular?

2. Solve the following system without multiplying $\mathbf{LU}$:

$$\mathbf{LUx} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \\ 2 \end{Bmatrix}$$

3. If the inverse of the matrix $\mathbf{A}^2$ is $\mathbf{B}$, show that the inverse of $\mathbf{A}$ is $\mathbf{AB}$. (Thus, $\mathbf{A}$ is invertible whenever $\mathbf{A}^2$ is invertible.)

4. For the following system, what is the triangular system after elimination, and what is the solution?

$$\begin{array}{rcrcrcr} x & + & 2y & - & 4z & = & -4 \\ 5x & + & 11y & - & 21z & = & -22 \\ 3x & - & 2y & + & 3z & = & 11 \end{array}$$

5. Determine the values of k so that the following system in unknowns x, y, and z has (a) a unique solution, (b) no solution, and (c) an infinite number of solutions.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & kz & = & 3 \\ x & + & ky & + & 3z & = & 2 \end{array}$$

ApSc 212 – Assignment 2

Write and verify a general purpose Gaussian elimination subroutine (in the language of your choice) with the following calling sequence (expressed in Fortran format):

`CALL SOLVE(A,B,N,IDIM,NR,IFLAG)`, where

`A(I,J)`	=	matrix entry in row `I`, column `J` of the square nonsymmetric matrix `A` (input/output).
`B(I,J)`	=	`I`th component of `J`th right-hand side (input/output),
`N`	=	order of system (input),
`IDIM`	=	row-dimension of arrays `A` and `B` (input),
`NR`	=	number of right-hand side vectors (input),
`IFLAG`	=	flag to indicate if the input matrix is already in upper triangular form with the multipliers in the lower triangle so that the elimination can be skipped (input): 0 = first RHS for this matrix, 1 = subsequent RHS for this matrix.

On return, `B` contains the solution, and, if `IFLAG=0`, `A` contains the upper triangular form in the upper triangle and the Gaussian elimination row multipliers in the lower triangle below the diagonal. If `IFLAG=1`, `A` is not changed by this routine, and only the forward-backward substitution is performed.

Describe how you verified your program.

Solve $\mathbf{Ax} = \mathbf{b}$, where the square matrix $\mathbf{A}$ of order $n = 10$ is defined as $A_{ij} = n - |i - j|$, and the $n \times 1$ right-hand side $\mathbf{b}$ is given by $b_i = 3[n(n-1)/2 + i(n - i + 1)]$. Also submit a listing of your subroutine and a printout of matrix $\mathbf{A}$ as it exists on return from the subroutine.

ApSc 212 – Assignment 3

1. Prove that the transpose of a matrix product is equal to the product of the individual transposes in reverse order. Hint: Use the definition of matrix multiplication in index notation.

2. Prove that the inverse of a triangular matrix is triangular. Hint: Consider $\mathbf{Ux} = \mathbf{e}^{(i)}$, where $\mathbf{U}$ is an upper triangular matrix, and $\mathbf{e}^{(i)}$ is the ith column of the identity matrix.

3. Prove that, for a matrix $\mathbf{A}$, $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$. Hint: Use the definition of the matrix inverse.

4. Describe the column space and the null space of the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Decide whether or not the following vectors are linearly independent: $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (1, 0, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1, 1)$, $\mathbf{v}_4 = (0, 1, 0, 1)$. Decide also if they span $\mathbb{R}^4$.

6. Express $\mathbf{v} = (1, -2, 5)$ in $\mathbb{R}^3$ as a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, where $\mathbf{u}_1 = (1, -3, 2)$, $\mathbf{u}_2 = (2, -4, -1)$, $\mathbf{u}_3 = (1, -5, 7)$.

7. Express the polynomial $v = t^2 + 4t - 3$ as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, and $p_3 = t + 3$.

ApSc 212 – Assignment 4

1. Sketch the effect that the transformation matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

has on the unit square $(0 < x < 1,\ 0 < y < 1)$ in the $x - y$ plane. How would you describe this transformation?

2. What 3×3 matrices represent the transformations that

   (a) project every vector onto the xy-plane?

   (b) reflect every vector through the xy-plane?

   (c) rotate objects 90° (positive by the right-hand rule) about the z-axis?

   (d) rotate objects sequentially about the z, y, and x axes (in that order), all by 90°?

   (e) carry out the same three rotations as in (d), but through 180°?

   Consider each rotation to be with respect to a coordinate system fixed in space.

3. On the space P_3 of cubic polynomials, what 4×4 matrix represents the second derivative d^2/dt^2? Construct that matrix from the standard basis 1, t, t^2, t^3. What is its null space, what is its column space, and what do these spaces mean in terms of polynomials?

4. Use elimination to find the triangular factors $\mathbf{A} = \mathbf{LU}$, if

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Under what conditions on the numbers a, b, c, d are the columns of $\mathbf{A}$ linearly independent?

ApSc 212 – Assignment 5

1. Find a vector $\mathbf{w}$ orthogonal to the row space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

2. Find all vectors which are perpendicular to the vectors $(1, 4, 4, 1)$ and $(2, 9, 8, 2)$.

3. Consider the vector space $P_n(t)$ of polynomials in t of degree $\leq n$. Determine whether or not $1 + t$, $t + t^2$, $t^2 + t^3$, $\ldots$, $t^{n-1} + t^n$ form a basis of $P_n(t)$.

4. Let V be the vector space of real 2×2 matrices. Determine whether the following set of matrices forms a basis for V.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

ApSc 212 – Assignment 6

1. Prove that the trace of the projection matrix $\mathbf{P} = \mathbf{a}\mathbf{a}^T/(\mathbf{a}^T\mathbf{a})$ is unity. (The trace of a matrix is defined as the sum of its diagonal entries.)

2. Prove that the following quantities are invariant under an orthogonal coordinate transformation:

   (a) the length of a vector

   (b) the trace of a matrix (the sum of the diagonal entries)

   (c) the inner product of two vectors

3. Consider the matrix $\mathbf{A}$ of order $n = 5$, where $A_{ij} = n - |i - j|$.

   (a) Find the LU decomposition of $\mathbf{A}$, where $\mathbf{L}$ is a lower unit triangular matrix, and $\mathbf{U}$ is an upper triangular matrix. How do you know your answer is correct? Is this decomposition unique?

   (b) What is $\mathbf{A}^{-1}$? How did you find it?

   (c) What is $\det(A)$? How did you compute it?

4. Given the four points $(0.5, 5.43)$, $(1, 4.47)$, $(1.5, 3.89)$, and $(2.5, 3.33)$, find the least squares approximation to these data using the pair of basis functions 1 and e^{-x}.

5. Given the four points $(1, 10)$, $(2, 5)$, $(4, 2)$, and $(6, 1)$,

   (a) Find the least squares quadratic polynomial which approximates these data.

   (b) Using the pair of basis functions 1 and $1/x$, find the least squares approximation to these data.

   (c) For each of these two approximations, compute the residual R which results. Which is the better approximation?

ApSc 212 – Assignment 7

1. The Hilbert matrix is the square matrix $A_{ij} = 1/(i + j - 1)$. Using your equation solver (in double precision), solve the system $\mathbf{Ax} = \mathbf{b}$ for matrix orders $n = 4$, 6, 8, 10, 12, and 14, where $\mathbf{b}$ is specified so that the solution $\mathbf{x}$ is a vector of ones (i.e., $x_i = 1$ for all i). Display the solution $\mathbf{x}$ with the same precision with which it was computed (about 14 decimal digits).

2. Factor the following matrix into QR, recognizing that the first column is already a unit vector:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$$

3. Find an orthonormal basis for $\mathbb{R}^3$ starting with the vector $(1, 1, -1)$. Is this basis unique?

4. Find the trigonometic Fourier series representing the function $f(x) = e^x$ in the interval $(-\pi, \pi)$. Plot the function and its series representation for different numbers of terms (e.g., $N = 2$, 5, 10, 20, 40, where N is the upper limit on the summations). For these plots, use a fine resolution on x, and plot curves without symbols.

ApSc 212 – Assignment 8

1. For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix},$$

find the characteristic equation, all eigenvalues and eigenvectors, and an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is diagonal.

2. Find the eigenvalues and the diagonalizing matrix $\mathbf{S}$ for

$$\begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}.$$

3. If matrix $\mathbf{A}$ has eigenvalues 0 and 1, corresponding to the eigenvectors $(1, 2)$ and $(2, -1)$, respectively, how can you tell in advance that $\mathbf{A}$ is symmetric? What are $\mathbf{A}$'s trace and determinant? What is $\mathbf{A}$?