The George Washington University
Washington, D.C.

MAE 207/CE 221 – Theory of Elasticity

Spring 2010 – Main Campus

References: *Elasticity*, third edition, by J.R. Barber (Springer, 2010), ISBN 9789048138081.

Notes: *Elasticity* by G.C. Everstine

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Schedule: Wednesdays, Jan. 13 – May 5, 6:10 p.m. – 8:40 p.m.
No class: March 17, April 28 (designated Monday)
Mid-Term Exam: March 10
Final Exam: May 5

Description: Introduction to Cartesian tensors; deformation, stress, constitutive relations for linear elasticity; formulation of boundary value problems; variational principles; torsion and bending of prismatic rods; plane problems.

Objectives: To understand the basic equations of linear elasticity; to appreciate the variety of methods used to solve elasticity problems; to apply the fundamental equations by solving elementary elasticity problems.

Grading: Assignments 1/3, mid-term exam 1/3, final exam 1/3. All graded work must be completed in accordance with the GW Code of Academic Integrity (`http://www.gwu.edu/~ntegrity/code.html`). Students are encouraged to discuss the meaning of assignments and general approaches and strategies for handling those assignments, but it is not acceptable to share solutions.

Course Outline

1. Mathematical preliminaries; vectors; index notation; summation convention; volume; change of basis; orthogonal transformations; tensors; divergence theorem
2. Analysis of strain; deformation; general infinitesimal deformation; inhomogeneous deformations; infinitesimal strain tensor; compatibility; principal axes; finite deformation
3. Analysis of stress; body and surface tractions; stress tensor; equations of equilibrium; coordinate transformations; principal stresses; conservation of momentum and angular momentum
4. Equations of elasticity; Hooke's law; strain energy; material symmetry
5. Simplest problems of elastostatics; simple shear; simple tension; uniform compression; engineering elastic constants; stress and strain deviators; stable reference states
6. Boundary value problems in elastostatics; uniqueness
7. Torsion of circular and non-circular shafts; warping function; uniqueness and existence of warping function; harmonic functions; numerical analogue; stress function; torsion of elliptical cylinder; torsion of rectangular bars using warping and stress functions
8. Two-dimensional problems; plane strain; plane stress; stress compatibility equations; Airy stress function; biharmonic equation; bending of beams

MAE 207/CE 221 – Assignment 1

1. The vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ is referred to as the *position vector* in Cartesian coordinates. Find $\nabla \cdot \mathbf{r}$ and $\nabla \times \mathbf{r}$.

2. Show that if S_{ij} is symmetric and A_{ij} is antisymmetric, then $S_{ij}A_{ij} = 0$. Express this result in matrix notation.

3. Substitute $u_i = B_{ij}v_j$ and $C_{ij} = p_i q_j$ into $w_i = C_{ij}u_j$, and transcribe the resulting equation into direct (vector-operator) notation.

4. Evaluate the surface integral of the normal component of the vector $\mathbf{F} = x_i\mathbf{e}_i$ over the closed surface of the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

5. Evaluate the surface integral of the normal component of the vector field

   $$\mathbf{F} = yz\mathbf{e}_x + xz\mathbf{e}_y + xy\mathbf{e}_z$$

   over the closed surface of the 3-D region bounded above by $z = 1$ and below by $z = x^2 + y^2$.

6. A uniform (constant) pressure is applied to the surface of a body of arbitrary shape. Prove that the resultant moment $\mathbf{M}$ of this distribution of surface loading is zero.

7. An arbitrary body of volume V is fully submerged in a fluid of uniform weight density γ. Given that the hydrostatic pressure p is γz at a fluid depth z (positive down), compute the resultant force $\mathbf{F}$ acting on the body.

8. Given a general closed surface S for which the position vector $\mathbf{r} = x_i\mathbf{e}_i$ and normal $\mathbf{n}$ are known at every point, derive a formula for the volume enclosed by S. Verify your relation for the special case of a sphere.

MAE 207/CE 221 – Assignment 2

1. Find the change in volume of a body of arbitrary shape, initially of volume V_0, due to the displacement field $u_i = cx_i$, where c is an infinitesimal constant.

2. Consider the simple shearing deformation $u_1 = kx_2$, $u_2 = u_3 = 0$, where k is an infinitesimal constant. Calculate the components of (a) the displacement gradient matrix $u_{i,j}$, (b) the infinitesimal strain tensor ε_{ij}, and (c) the rotation ω_{ij}. (d) Find the principal axes and principal strains.

3. An elastic solid is heated nonuniformly with a temperature distribution $T(x, y, z)$, where T is a function of x, y, z. If each element in the body has unrestrained thermal expansion, the strain components will be $\varepsilon_{ij} = \alpha T \delta_{ij}$, where α is the (constant) coefficient of thermal expansion. Prove that this situation can occur only when T is a linear function of x, y, z. Hint: For a function $H(x, y, z)$, $\partial H/\partial z = 0$ implies that $H = f(x, y)$.

4. A parallelepiped occupies the domain $0 \leq x \leq L$, $-h \leq y \leq h$, $-b \leq z \leq b$. It is deformed in such a manner that a point $P(x, y, z)$ is displaced to $\bar{P}(\bar{x}, \bar{y}, \bar{z})$, where

   $$\bar{x} = (c - y)\cos(x/c), \quad \bar{y} = (c - y)\sin(x/c), \quad \bar{z} = z,$$

   where c is a constant. Determine the restrictions which must be placed on c in order that the displacement may be continuously possible.

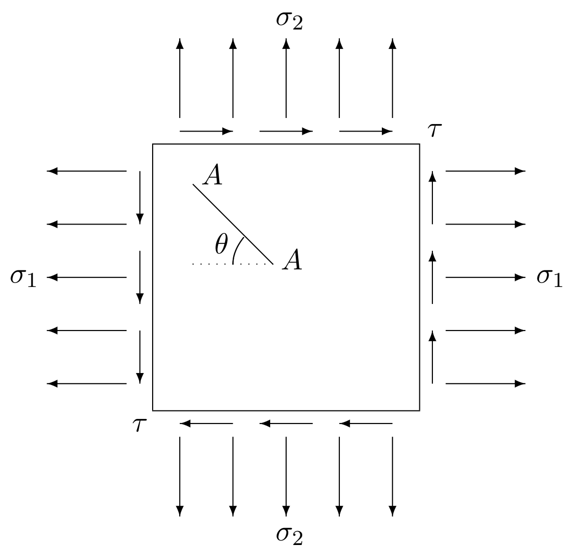
5. For the two-dimensional small displacement theory, the strains for a cantilever beam of length L and depth $2a$ subjected to a concentrated lateral load P at the free end are given by

   $$\varepsilon_{xx} = Axy, \quad \varepsilon_{yy} = -\nu Axy, \quad 2\varepsilon_{xy} = A(1 + \nu)(a^2 - y^2),$$

   where A and ν are constants, and the origin of coordinates is at the center of the free end (with the x-axis the central longitudinal axis of the beam, and the y-axis coincident with the force P). Show that continuous single-valued displacements (u, v) are possible. With the boundary conditions $u(L, 0) = v(L, 0) = u_{,y}(L, 0) = 0$, derive formulas for (u, v) as functions of (x, y).

MAE 207/CE 221 – Assignment 3

1. A square plate is loaded as shown below. Compute the shear stress and the tensile stress on section A-A in terms of σ_1, σ_2, τ, and θ. Specialize the result for $\sigma_1 = \sigma_2$ and $\theta = 45°$.



2. The stress tensor for some problem is

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix}.$$

Determine the normal stress and shear stress on the plane whose normal is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

3. The stress state at a point is defined by the stress components $\sigma_{xx} = a$, $\sigma_{xy} = b$, $\sigma_{yy} = c$, $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. Determine the stress components relative to a new set of axes obtained by a rotation of 30° about the y-axis.

4. The stress tensor is

$$\begin{bmatrix} 2 & 0 & \sqrt{3} \\ 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}.$$

Determine the principal stresses and the direction cosines of the normal to the plane on which $\sigma_{\max}$ acts.

MAE 207/CE 221 – Assignment 4

1. The following stress array $\sigma_{ij}(x,y)$ is proposed as a solution to a certain *equilibrium* problem of a plane body bounded in the domain $-L/2 \leq x \leq L/2$, $-h/2 \leq y \leq h/2$:

$$\sigma_{xx} = Ay + Bx^2y + Cy^3,\ \sigma_{yy} = Dy^3 + Ey + F,\ \sigma_{xy} = (G + Hy^2)x,\ \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0,$$

where A, B, C, D, E, F, G, H are constants.

(a) Determine the conditions under which this array is a possible *equilibrium* solution.

(b) It is proposed that the domain be loaded such that

$$\sigma_{xy}(x, \pm h/2) = 0,\ \sigma_{yy}(x, h/2) = 0,\ \sigma_{yy}(x, -h/2) = -\sigma = \text{constant},\ \sigma_{xx}(\pm L/2, y) = 0.$$

Determine whether the proposed stress array satisfies these conditions.

2. Let r be the length of the position vector $\mathbf{r} = x_i\mathbf{e}_i$. Thus, $r^2 = x_ix_i$. Show that $r_{,i} = x_i/r$. Interpret this formula in vector form.

3. A spherical hole of radius R in an infinite body of isotropic elastic material contains gas under the pressure p_0. Assuming a displacement field of the form $u_i = cr^kx_i$, where $r^2 = x_ix_i$, determine the constants c and k. Show that the stress on any surface facing the radial direction is purely normal, and calculate this normal stress. Show that the stress on any surface element with normal perpendicular to the radial direction is purely normal, and calculate this "hoop" stress. Show that the hoop stress is greatest at the hole. (It is not necessary to use any coordinate system other than Cartesian to solve this problem.) Hint: Compute strain and stress, enforce equilibrium, and impose boundary conditions.

MAE 207/CE 221 – Assignment 5

1. Determine whether or not the following stress components are a possible solution of an elasticity problem for an isotropic material in the absence of body forces and temperature effects:

$$\sigma_{xx} = ayz,\ \sigma_{yy} = bxz,\ \sigma_{zz} = cxy,\ \sigma_{xy} = dz^2,\ \sigma_{xz} = ey^2,\ \sigma_{yz} = fx^2,$$

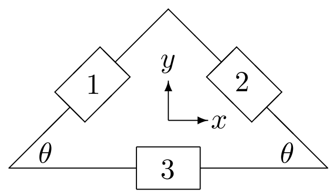
   where a, b, c, d, e, f are constants.

2. Consider a rectangular plate in the x-y plane subjected to a uniform stress σ in the x direction along the edges parallel to the y axis. The coordinate origin is at the center of the plate. For an isotropic, homogeneous elastic material, derive expressions for the displacements $u(x, y), v(x, y)$.

3. An isotropic spherical shell of internal radius R_0 and external radius R_1 is subjected to internal and external pressures p_0 and p_1, respectively. Assuming a displacement field of the form $u_i = cr^k x_i$, where $r^2 = x_i x_i$, find the displacement field and the stress. Notice that the stress is independent of the elastic constants. Calculate the radial and hoop stresses. Find the conditions under which the hoop stress at the interior surface is compressive rather than tensile.

MAE 207/CE 221 – Assignment 6

1. A strain gage rosette is attached to a point of the free unloaded surface of a linearly elastic isotropic body. Under deformation of the body, the strain gages in arms 1, 2, 3 record direct strains ε_1, ε_2, and ε_3, respectively. Derive expressions for all six Cartesian components of strain and stress (ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy}, ε_{xz}, ε_{yz}, σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{xz}, σ_{yz}) at the point in terms of ε_1, ε_2, ε_3, θ, and the engineering constants E, G, ν.



2. A beam of rectangular cross-section (with surfaces $x = 0, L$, $y = \pm h$, $z = \pm w$) is bent by tractions $(Ty/h)\mathbf{e}_1$ on $x = L$ and tractions equal in magnitude but oppositely directed on $x = 0$, where T is a constant. The remaining surfaces are free from tractions, and there is no body force.

   (a) What is the simplest three-dimensional stress distribution, consistent with these tractions, that you can think of?

   (b) Verify that this distribution satisfies the equilibrium equations (or guess again).

   (c) Compute the associated strains (for an isotropic material), and verify that these strains satisfy the compatibility conditions (or guess again, if they do not).

   (d) Find the displacement field. Is the displacement field unique?

MAE 207/CE 221 – Assignment 7

1. Show that cylinders with circular cross-sections are the only bodies whose lateral surface can be free from external load when the stress components due to torsion about the z axis are given by

$$\boldsymbol{\sigma} = \mu\alpha \begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}.$$

   Hint: Integrate to obtain the stress function.

2. Show that the function

$$\psi = \frac{\mu\alpha}{2}\left[b^2 - r^2 + 2a(r^2 - b^2)\frac{\cos\theta}{r}\right]$$

   is the stress function for torsion of a circular shaft of radius a with a circular groove of radius b. This function is expressed in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$. Derive formulas for σ_{zx} and σ_{zy}, and express in polar coordinates. How would you determine the location of the maximum shear stress?