The George Washington University
Washington, D.C.

MAE 286 – Numerical Solution Techniques in Mechanical and Aerospace Engineering

Fall 2009 – Main Campus

References: *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, third edition, by G.D. Smith (Oxford University Press, 1985), ISBN 0198596502; *The Finite Element Method for Engineers*, fourth ed., by K.H. Huebner, D.L. Dewhirst, D.E. Smith, and T.G. Byrom (Wiley, 2001), ISBN 0471370789.

Notes: *Numerical Solution of Partial Differential Equations* by G.C. Everstine

Instructor: Gordon C. Everstine, `http://gwu.geverstine.com`
geversti@gwu.edu, 301-977-0936

Schedule: Mondays, Aug. 31 – Dec. 14, 6:10 p.m. – 8:40 p.m.
Mid-Term Exam: Oct. 26
Final Exam: Dec. 14

Description: Development of finite difference and finite element techniques for solving elliptic, parabolic, and hyperbolic partial differential equations. Prerequisite: ApSc 213 or equivalent.

Objectives: To understand the fundamentals of finite difference and finite element solution of partial differential equations of engineering; to increase knowledge of the nature of solutions of equations of different types; to apply the numerical algorithms by solving various equations numerically.

Grading: Assignments 1/3, mid-term exam 1/3, final exam 1/3. All graded work must be completed in accordance with the GW Code of Academic Integrity (`http://www.gwu.edu/~ntegrity/code.html`). Students are encouraged to discuss the meaning of assignments and general approaches and strategies for handling those assignments, but it is not acceptable to share solutions and computer codes.

Course Outline

1. Ordinary differential equations; Euler's method; truncation error; Runge-Kutta methods; systems of equations; boundary value problems; finite differences; shooting methods
2. Classical equations of mathematical physics (Laplace, Poisson, wave, Helmholtz, heat); classification of PDEs; examples; transformation to non-dimensional form
3. Finite difference solution of parabolic equations; explicit and implicit methods; Crank-Nicolson method; stability
4. Finite difference solution of hyperbolic equations; domain of dependence; stability
5. Finite difference solution of elliptic equations; direct and iterative solution; derivative boundary conditions
6. Direct finite element analysis; spring, truss, and beam systems; matrix partitioning and constraints; 2-D continuum problems; change of basis
7. Calculus of variations; the brachistochrone; constraints
8. Variational principles; index notation and summation convention; deriving variational principles; shape functions; compatibility; element matrices; method of weighed residuals (Galerkin's method)
9. Potential fluid flow; symmetry; free surface flows; 2-D wave maker; variational principle; mechanical analogy

MAE 286 – Assignment 1

1. Consider the ordinary differential equation

$$y' = xy^{1/3}, \; y(1) = 1,$$

whose solution is $y = [(x^2 + 2)/3]^{3/2}$.

(a) For $1 \le x \le 5$, compute numerical solutions to this equation using Euler's method with step sizes $h = 0.10$, 0.05, and 0.01. Compare with the exact solution, and comment on the change in error as h decreases.

(b) For $1 \le x \le 5$, compute a numerical solution to this equation using the second-order Runge-Kutta approach with $h = 0.10$, and comment on its error.

2. Solve the ordinary differential equation

$$y' = x^2 + x - y, \; y(0) = 0,$$

by the second-order Runge-Kutta method to find an approximation to y for $0 \le x \le 1$ using step sizes $h = 0.2$, 0.1, and 0.05. Compare with the exact solution

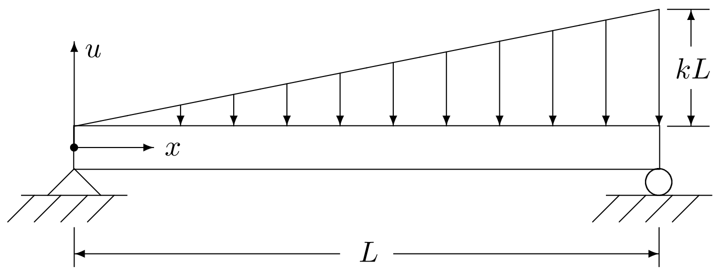
$$y = -e^{-x} + x^2 - x + 1,$$

and comment on the change in error as h decreases.

3. Obtain a linear algebraic equation solver in the programming language of your choice, and verify that you are able to solve algebraic equations of the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a real $n \times n$ matrix, $\mathbf{b}$ is a known $n \times 1$ vector (real), and $\mathbf{x}$ is an unknown $n \times 1$ solution vector. Submit a description of what you did and a printout of your test program.

MAE 286 – Assignment 2

A simply-supported slender beam of length L is subjected to a uniformly varying load per unit length $F(x) = kx$ as shown below, where the constant k has the units of stress.



a. Using elementary beam theory, derive the second-order ordinary differential equation and boundary conditions which describe the vertical deflection $u(x)$ of the neutral axis of the beam.

b. For the numerical solutions in Parts (b) and (c), assume $L = 1$, and $k = 360\, EI/L^4$, where E is Young's modulus, and I is the cross-sectional moment of inertia. Solve the system derived in Part (a) numerically using central finite differences. Use the programming language of your choice and any available equation solver to solve the system of linear algebraic equations which results. Solve the system for meshes $h = 0.2$, 0.1, 0.05, 0.025, and 0.0125.

c. Solve the system derived in Part (a) numerically using the shooting method based on the Euler method with $h = 0.02$.

MAE 286 – Assignment 3

1. Use the forward-difference explicit method to solve the heat equation

$$u_t(x,t) = u_{xx}(x,t),\ \ 0 < x < 1,\ \ 0 < t < 0.4,$$

with the initial condition $u(x,0) = 4x - 4x^2$ and the boundary conditions $u(0,t) = u(1,t) = 0$. Use the following step sizes:

(a) $h = \Delta x = 0.2$ and $k = \Delta t = 0.02$

(b) $h = \Delta x = 0.2$ and $k = \Delta t = 0.033333$

Tabulate results for all (x,t), and comment on the two sets of results.

2. Use the Crank-Nicolson method to solve the heat equation

$$u_t(x,t) = u_{xx}(x,t),\ \ 0 < x < 1,\ \ 0 < t < 0.15,$$

with the initial condition $u(x,0) = \sin \pi x + \sin 3\pi x$ and the boundary conditions $u(0,t) = u(1,t) = 0$. Use the step sizes $h = \Delta x = 0.1$ and $k = \Delta t = 0.01$. Tabulate results for all (x,t).

MAE 286 – Assignment 4

Use the explicit finite difference method to solve the wave equation for a vibrating string

$$u_{tt}(x,t) = 4u_{xx}(x,t),\ 0 < x < 1,\ 0 < t < 1,$$

with the boundary conditions

$$u(0,t) = 0,\ u(1,t) = 0$$

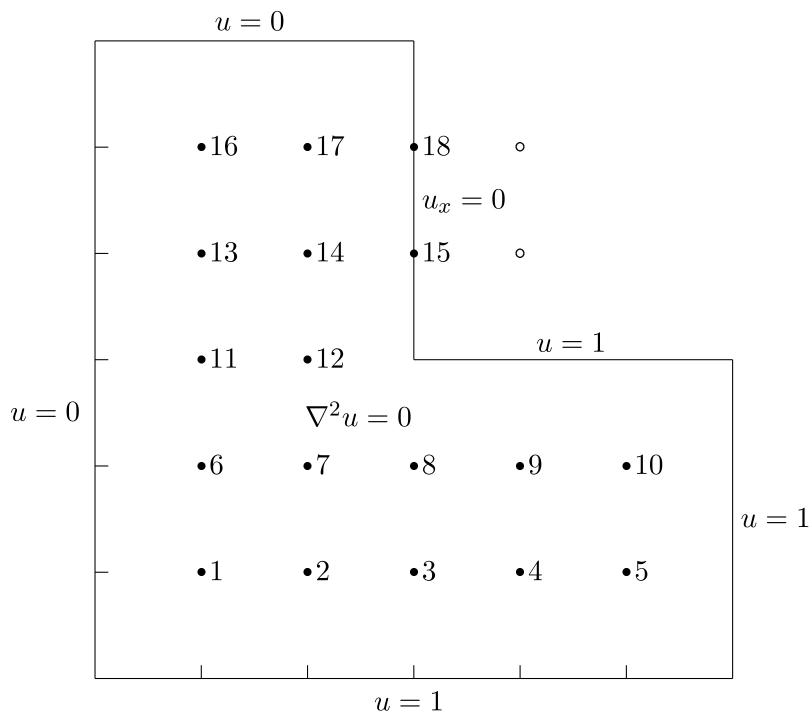
and initial conditions

$$u(x,0) = f(x) = \sin \pi x + \sin 2\pi x,\ u_t(x,0) = g(x) = 0.$$

Choose $h = \Delta x = 0.1$ and the largest value of $k = \Delta t$ consistent with stability. Tabulate results for all (x,t). Can you infer the period of the solution from these results?

MAE 286 – Assignment 5

Consider the boundary value problem defined in the figure below. With $h = 1/3$ and the finite difference mesh suggested, write the difference equations centered at Points 7, 18, and 5. Set up and solve the rest of the system.



Suggested approach:

Since manual setup of the equations could be very tedious, I would suggest using a simple computer program which reads a data file with one line for each mesh point. Each line would have the six variables

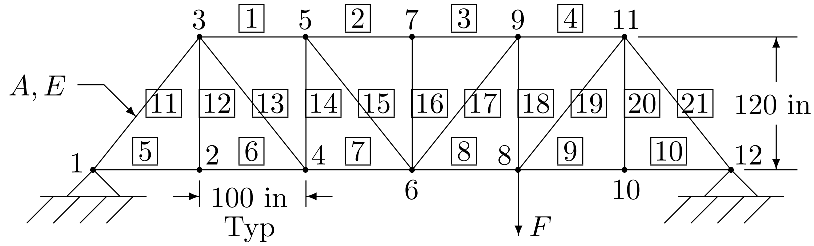
$$P, N_1, N_2, N_3, N_4, B,$$

where P is a point ID (1-18), N_i is the point ID of a neighbor (zero if that neighbor has a specified boundary value), and B is the sum of the specified boundary neighbor values for which $N_i = 0$. The program would read the data file, and build the matrix automatically. For each P, 4 is placed on the diagonal, and B is placed on the RHS. For each nonzero N_i, -1 is added to the off-diagonal, thus allowing for the double weighting associated with derivative boundary conditions. (That is, some points could be listed twice. For example, in Eq. 3.68 of the notes, the corresponding data line would be "$18, 14, 14, 17, 19, 2hg_{18}$", with 14 listed twice. Each time 14 is read, a -1 would be added to the coefficient matrix.) This approach yields a reasonably convenient table-driven program. Table-driven approaches are often preferred, since they permit the solution of a variety of problem geometries and sizes with the same executable program. This suggested approach (which is not required) allows one to solve Laplace's equation for arbitrary 2-D geometry with either a Dirichlet or Neumann (derivative) boundary condition at each point.

Notice that, because of a maximum principle associated with the Laplace equation, the extrema must occur on the boundary. Thus, for this problem, $0 \le u(x, y) \le 1$. Also, Point 1 is expected to have a solution approximately equal to 0.5, the average of the nearby boundary values.

MAE 286 – Assignment 6

Develop a computer program to compute displacements and stresses in reasonably arbitrary 2-D pinned truss structures. The truss members may be assumed to be made of the same material. Test the program by calculating the displacements and stresses for the pinned truss structure shown below. Assume $E = 30 \times 10^6$ psi and $A = 35$ in^2 for each truss member, and $F = 10^5$ lb. Submit also a listing of your program.



Suggested program flow:

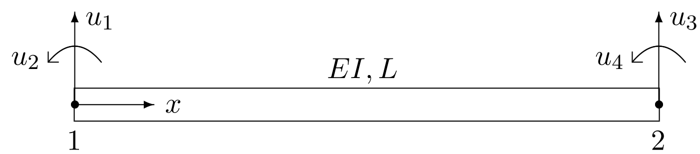
1. Read from data and store the following:
   (a) number of grid points (ng), number of elements (ne), number of points with nonzero applied force (nf), number of DOF with constraint (nc)
   (b) Young's modulus E
   (c) grid point coordinates: $x(i), y(i)$, $i = 1, 2, \ldots, ng$
   (d) element connectivity: grid point 1 $g1(i)$, grid point 2 $g2(i)$, area $A(i)$, $i = 1, 2, \ldots, ne$
   (e) forces: grid point $fg(i)$, x-component $fx(i)$, y-component $fy(i)$, $i = 1, 2, \ldots, nf$
   (f) constraints: grid point $cg(i)$, component $cc(i)$, value $cv(i)$, $i = 1, 2, \ldots, nc$
2. Assemble unconstrained stiffness matrix $\mathbf{K}$ and force vector $\mathbf{F}$.
3. Modify $\mathbf{K}$ and $\mathbf{F}$ due to constraints. Use of the large spring method is OK.
4. Solve $\mathbf{Ku} = \mathbf{F}$ for displacement vector $\mathbf{u}$.
5. Compute stresses. Note that, for small displacements, if $L^2 = a^2 + b^2$, $L\Delta L \approx a\Delta a + b\Delta b$, in which case $\sigma = E(a\Delta a + b\Delta b)/L^2$.

Some ways to verify your program:

1. Solve a problem with a known solution.
2. Apply a symmetric load, and check for a symmetric solution.
3. Look for truss members with zero load.
4. Remove enough constraints to cause an ill-posed problem.

MAE 286 – Assignment 7

According to elementary beam theory, the transverse displacement w for a beam in flexure is related to the bending moment M by $M(x) = EIw''(x)$, where E is the Young's modulus of the beam material, and I is the centroidal moment of inertia of the beam cross section. Derive the 4×4 stiffness matrix for a beam in flexure in 2-D. DOF u_2 and u_4 shown below are the rotations (slopes) and have the dimensions of radians. The corresponding generalized forces are moments. Use the direct approach, in which the stiffness matrix term K_{ij} is defined as the (generalized) force at DOF i corresponding to a unit (generalized) displacement at DOF j with all other displacement DOF held fixed.

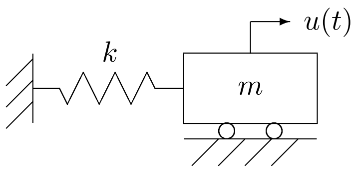


MAE 286 – Assignment 8

1. Consider the mass-spring system shown below, for which the kinetic energy $T = \frac{1}{2}m\dot{u}^2$ and potential energy $V = \frac{1}{2}ku^2$, where m is mass, k is stiffness, u is displacement from the equilibrium, and $\dot{u}$ is velocity. In Hamiltonian mechanics, the path $u(t)$ followed by the mass is that which minimizes the energy integral

$$\int_{t_1}^{t_2} (T - V)\, dt,$$

where $L = T - V$ is called the Lagrangian of the system. Find the equation of motion of the mass. That is, what differential equation does $u(t)$ satisfy?



2. Find the curve C of fixed length L which encloses maximum area. Hint: According to a variant of the divergence theorem in two dimensions, the area bounded by C is given by

$$A = \frac{1}{2} \oint_C (x\, dy - y\, dx).$$

MAE 286 – Assignment 9

Develop a finite element computer program to solve Laplace's equation $\nabla^2\phi = 0$ in arbitrary two-dimensional domains with Dirichlet boundary conditions. Use the three-node linear triangular element. The program should be able to compute both the potential function ϕ at each grid point and the gradient $\nabla\phi$ of the potential function for each element. Use the program to solve Laplace's equation in the equilateral triangle having altitude $a = 6$ and boundary conditions

$$\phi = \begin{cases} (9y^2 + a^2)/18 & \text{on } x = -a/3 \\ 2(9x^2 - 3ax + a^2)/27 & \text{on } y = \pm(x - 2a/3)/\sqrt{3} \end{cases}$$

Exploit symmetry by modeling only the upper half of the domain as shown; the symmetry condition on $y = 0$ is $\partial\phi/\partial n = 0$.