Symmetry

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28 April 2004

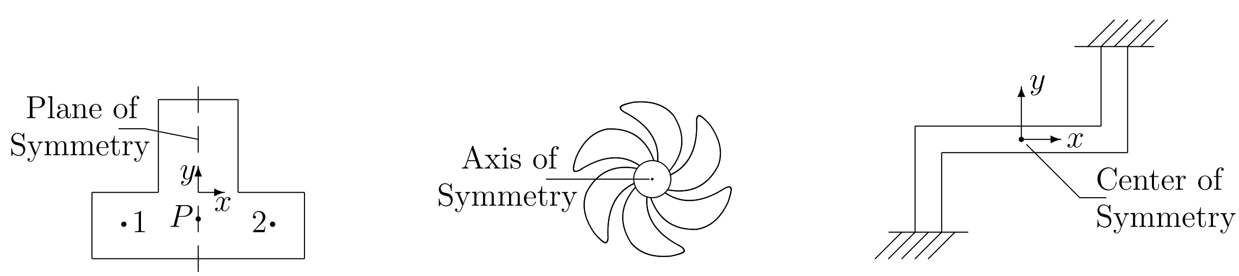
The analysis of engineering problems with complex geometry is tedius at best. However, for problems with symmetry, it is possible to gain some information about the solutions from the symmetry alone. Moreover, with symmetry present, the analyst needs to treat only a portion of the overall domain of interest. For example, a elastic structure possessing one plane of mirror symmetry can be analyzed by modeling only one-half of the structure, whether the loads are symmetric or not.

To be useful, symmetry must be exploited systematically and with confidence. Here we review and summarize the basic concepts involved in the systematic application of symmetry to elasticity problems, where *symmetry* refers to objects rather than physical laws or materials. In general, group theory is the mathematical language for discussions of symmetry (particularly in quantum mechanics), but such an approach is not necessary for elastic applications and will not be used here. Although the emphasis is this discussion will be applications in elasticity and structural mechanics, the same concepts carry over into other areas.

Types of Symmetry and Definitions

In engineering applications, the most commonly encountered types of symmetry are reflective (or mirror) symmetry, rotational (or axial) symmetry, and inversion symmetry. Examples of these three types are shown in Fig. 1.

An object possesses symmetry if the application to the object of some operation (such as a reflection, rotation, or inversion) transforms the object into an equivalent configuration.



(a) Reflective Symmetry (b) Rotational Symmetry (c) Inversion Symmetry

Figure 1: Examples of Different Types of Symmetry.

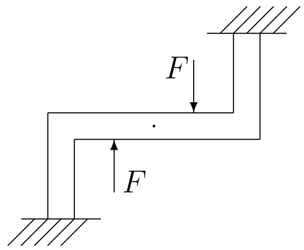


Figure 2: Structure With Symmetric Loads.

For engineering applications, this characterization of symmetry requires not only geometrical symmetry, but also symmetry with respect to material properties and restraints. For example, in Fig. 1b, if one propeller blade were made of aluminum and another of bronze (an unlikely situation), there would be no symmetry to exploit. In some situations, other properties may also play a role in deciding the presence of symmetry; for example, thermal radiation problems require symmetry with respect to color. Symmetry can normally be identified by inspection.

The characterization of symmetry in a particular situation is not necessarily unique. For example, the symmetry of Fig. 1c can also be characterized as a sequence of two reflections, one in the yz-plane followed by one in the xz-plane, or *vice versa*. The same structure could also serve as an example of a structure with rotational symmetry (with a rotation angle of $180°$).

In general, reflective symmetry is viewed as the fundamental type of symmetry, since it can be shown that all symmetric transformations of finite figures in three dimensions reduce to successive reflections in not more than three planes (which might not even be planes of symmetry).

Once the symmetry properties of a structure are identified, the loads can be addressed. The question of whether a given system of loads is symmetric depends on the structure to which that system is applied. Specifically, a system of loads, when applied to a structure possessing certain symmetry, is defined as *symmetric* if it is transformed into an equivalent configuration by the symmetry operations of the structure. The system of loads is defined as *antisymmetric* if the symmetry operations plus a negation of signs of all loads transform them into an equivalent configuration. For example, the loads in Fig. 2 are symmetric when applied to that stucture, whose symmetry is characterized by the sequence of two relections indicated above.

The Guiding Principle

The principle on which all applications of symmetry are based is that equivalent causes produce equivalent effects. In the context of elasticity and structural mechanics, the practical effect of this principle is that symmetric loads produce symmetric effects (displacements, stresses, etc.), and antisymmetric loads produce antisymmetric effects.

Boundary Conditions

When only a portion of a symmetric structure is modeled, the basic principle provides the tool for systematically deriving the symmetric or antisymmetric boundary conditions which must be applied at artificial boundaries introduced because of symmetry. Emphasis in this section will be restricted to planes of symmetry, since reflective symmetry has already been identified as the fundamental type of symmetry.

Consider, for example, the symmetric domain shown in Fig. 1a, where P denotes a typical point in the plane of symmetry. We define a Cartesian coordinate system with the x-direction normal to the plane of symmetry and the yz-plane parallel to the plane of symmetry. To derive the symmetric and antisymmetric boundary conditions to be applied at P if only half the domain is modeled, we use the following procedure:

1. Consider in turn each displacement component at that point.

2. Apply to that component (assumed to be nonzero) the symmetry (or antisymmetry) operations characterizing the structure.

3. Observe whether or not the component is transformed into itself.

If it is transformed into itself, the component may be nonzero and not violate symmetry. If, on the other hand, the component is *not* transformed into itself, the component must vanish in order not to violate symmetry. For reflective symmetry, the relevant symmetry operation is a reflection into the yz-plane containing P.

For example, assume that u_x, the x-component of displacement, is nonzero at P. The reflection produces an image of u_x with the opposite orientation. The addition negation of sign (for antisymmetry) yields a result coinciding with the original configuration. Therefore, u_x must vanish at P in order not to violate symmetry, but u_x may be nonzero for antisymmetric behavior. Similarly, we find that u_y and u_z vanish at P for antisymmetric behavior and may be nonzero for symmetric behavior.

In classical elasticity, the only displacement variables are the displacements themselves. However, the engineering approximations for beams, plates, and shells introduce additional variables for the slopes (or rotations) at the points. Thus, it is of interest to include rotational degrees of freedom in this discussion of symmetry.

Rotational degrees of freedom, if present, require slightly different treatment. Let Points 1 and 2 in Fig. 1a be image points of each other. Symmetric moments applied to these points must have opposite signs. For example, if the moment $M_z = 10$ lb-in is applied at Point 2, its symmetric counterpart is the moment $M_z = -10$ lb-in at Point 1. In other words, the reflection of an *axial vector* (a rotation or moment) into a plane requires an additional negation of sign compared with the way ordinary vectors reflect. The mathematical basis for this result is that reflection is an improper orthogonal transformation.

The application of the symmetry operation (reflection) to the rotational components R_x, R_y, and R_z at Point P in Fig. 1a indicates that, for symmetric behavior, R_y and R_z must vanish in order not to violate symmetry, and $R_x = 0$ for antisymmetry. To summarize, if the displacement degrees of freedom include both translations and rotations, the boundary conditions to impose at Point P are

$$\begin{cases} u_x = R_y = R_z = 0 & \text{for symmetry} \\ R_x = u_y = u_z = 0 & \text{for antisymmetry.} \end{cases} \tag{1}$$

The results expressed in these equations may be generalized as follows: Points lying in a plane of *symmetry* can suffer no translation out of the plane and no rotation about in-plane lines. For *antisymmetry*, the complementary set of degrees of freedom is constrained. The complementary nature of the symmetric and antisymmetric boundary conditions is a general result which follows from the observation that the only distinction between antisymmetry and symmetry is the additionl negation in the symmetry operations.

When higher-order derivatives are used as degrees of freedom (as sometimes occurs with certain specialized finite elements), additional symmetric and antisymmetric constraints are also required. Such constraints are most easily derived from the observation that u_x, the translational component of displacement normal to a symmetry plane, is either an odd or an even function of x, depending on whether the behavior is symmetric or antisymmetric, respectively. Similarly, the translational components of displacement parallel to a symmetry plane are even and odd in x for symmetric and antisymmetric behavior, respectively. For even functions of x, the odd-order derivatives with respect to x must vanish for symmetry, and the even-order derivatives with respect to x vanish for antisymmetry. Conversely, for odd functions of x, the even- and odd-order derivatives with respect to x vanish for symmetry and antisymmetry, respectively. As before, the set of degrees of freedom constrained as a consequence of symmetry is complementary to the set which is constrained as a consequence of antisymmetry, and *vice versa*. This result follows directly from the definitions, since the only real distinction between symmetry and antisymmetry is the additional negation that distinguishes even from odd functions. Thus, if one can determine for some problem the symmetry boundary conditions, the antisymmetry conditions consist of the complementary set of degrees of freedom. One consequence of this property is that, for finite element models, the total number of degrees of freedom arising from symmetric and antisymmetric models of a structure must equal the original number of degrees of freedom for the entire structure, disregarding symmetry.

The symmetric conditions for scalar field problems (e.g., heat conduction or potential fluid flow) are obtained as special cases of the preceding development. At a plane of symmetry, the normal derivative of the field variable vanishes. (In finite element analysis, this condition is a *natural* boundary condition.) For antisymmetry, the function itself must vanish at a plane of symmetry.

Nonsymmetric Loads

In general, most load systems applied to structures are neither symmetric nor antisymmetric, but nonsymmetric. However, any nonsymmetric system can always be uniquely decomposed into the sum of a symmetric and antisymmetric system of loads as, for example, in Fig. 3. Given arbitrary loads F_1 and F_2 at a Point 1 and its image, Point 2, the symmetric part of the load F_s and the antisymmetric part F_a are given by

$$F_s = \frac{1}{2}(F_1 + F_2), \quad F_a = \frac{1}{2}(F_1 - F_2). \tag{2}$$

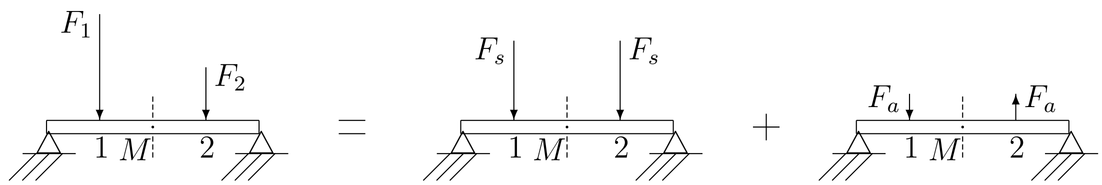


Figure 3: Decomposition of Nonsymmetric Loads.

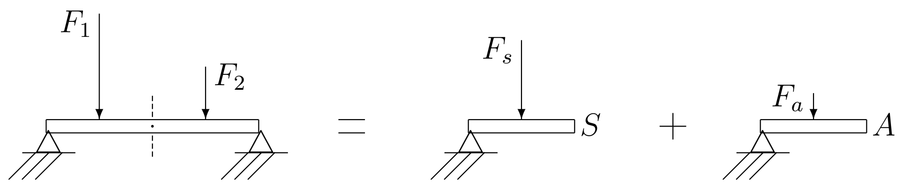


Figure 4: Superposition of Symmetric and Antisymmetric Solutions to Obtain Solution on Modeled Side.

In linear problems, to which the principle of superposition applies, only half the problem shown in Fig. 3 needs to be modeled. The analyst would model the left half, say, and solve the problem in two steps:

1. the symmetric part of the load is applied along with symmetric boundary conditions at the middle M, and
2. the antisymmetric part of the load is applied along with antisymmetric boundary conditions imposed at M.

Thus we have Fig. 4, for which the symmetric (S) and antisymmetric (A) boundary conditions are given in Eq. 1. Adding the two solutions in Fig. 4 yields the solution of the original problem only for the left side of the structure (the modeled side). To obtain the solution for the right side (the unmodeled side), the two solutions can be subtracted, as indicated in Fig. 5. Taking the difference of the symmetric and antisymmetric solutions has the practical effect of reversing the role played by the left and right sides. Thus, even though only the left side is modeled, the entire solution can be obtained.

Multiple Planes of Symmetry

The rectangular domain in Fig. 6 possesses two planes of symmetry (xz and yz). Hence the problem can be decomposed into four parts as shown. Any quadrant can be chosen to be modeled, and the four combinations of symmetric and antisymmetric boundary conditions

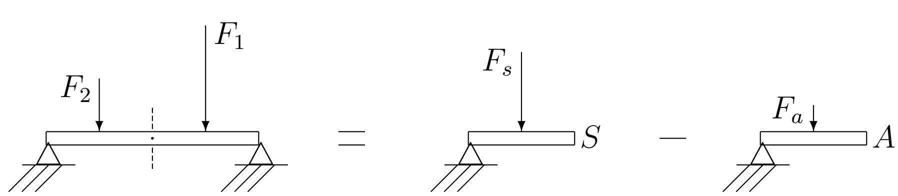


Figure 5: Superposition of Symmetric and Antisymmetric Solutions to Obtain Solution on Unmodeled Side.

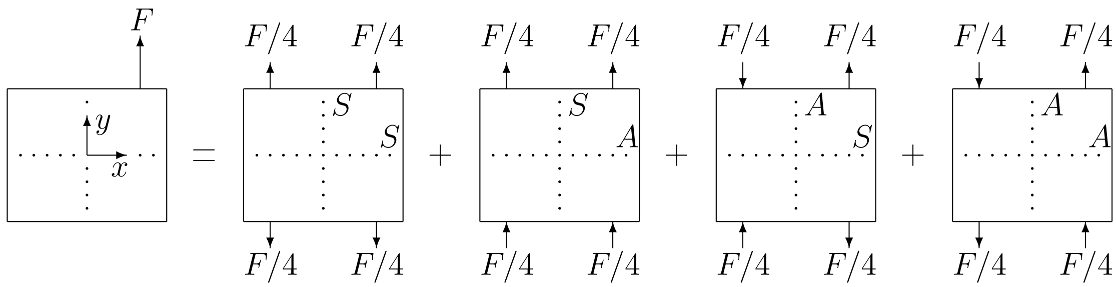


Figure 6: Multiple Planes of Symmetry.

(SS, SA, AS, and AA) imposed on the points lying in the two planes of symmetry. The four solutions can be combined in various ways to yield the solutions in all four quadrants.

Dynamics

The foregoing discussion has been devoted exclusively to statics problems, but free vibration problems (eigenvalue problems) can also exploit symetry. The calculation of all natural frequencies and mode shapes of a symmetric structure would require one eigenvalue analysis for each unique combination of symmetric and antisymmetric boundary conditions. For example, the natural frequencies of the domain of Fig. 6, which has two orthogonal planes of symmetry, can be otained by modeling only one quadrant and applying, in turn, each of the four combinations of boundary conditions.

All the preceding results for statics problems also apply to linear transient (time-dependent) situations, except that the entire history of time-dependent loads must be decomposed into symmetric and antisymmetric parts.